Social and Economic Networks

Introductory Lectures for BA Seminar

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Outline

Why study networks?

Applicable to a wide range of social and economic applications

- Information transmission about job opportunities
- Trade of goods and services
- Provision of informal insurance in developing countries
- Spread of innovations and diseases
- Voting behavior and opinion formation
- Peer effects in criminal activity and educational attainment
- Likelihood to succeed professionally

A friendship network at a US High School

Figure: Nodes coloured by student race; Currarini et al (2004)

The global financial network

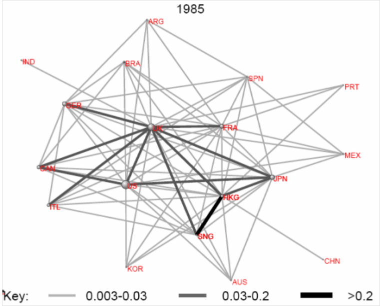


Figure: Haldane (2009)

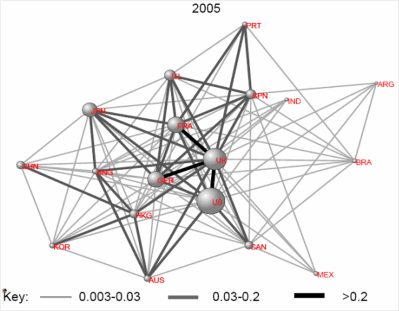


Figure: Haldane (2009)

Outline

Network representation

Defining a network as a graph (N, g)

Nodes Set $N = \{1, \ldots, n\}$ lists the *nodes* involved in network relations. Can represent people, firms, countries, etc.

Links Connections between nodes, described by *adjacency matrix* g of dimension *nxn* where g_{ij} describes connection from node i to node j

- Links can be directed (citations, web links) or undirected (family relations, alliances). If undirected $g_{ij} = g_{ji}$.
- Links can be weighted to account for strength of relationships (financial exposure). If unweighted $g_{ij} \in \{0, 1\}$.

Connectivity

Connections between nodes

Path
: A *path* between i and j is a sequence of links $i_1 i_2, i_2 i_3, ..., i_{K-1} i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, ..., K-1\}$, with $i_1 = i$ and $i_K = j$, and such that each node in the sequence distinct.

Geodesic
: A *geodesic* between nodes i and j is a shortest path between these nodes.

Connectivity

Connections between nodes

Walk A *walk* between i and j is a sequence of links $i_1 i_2, i_2 i_3, ..., i_{K-1} i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, ..., K-1\}$, with $i_1 = i$ and $i_K = j$.
NB: can cover the same node more than once.

Cycle A *cycle* is a walk $i_1 i_2, i_2 i_3, ..., i_{K-1} i_K$ that starts and ends at the same node and such that all other nodes are distinct.

Connectivity

Local connectivity

Neighbourhood
: The *neighbourhood* of a node i in network g, labelled $N_i(g)$ is the set of nodes linked to it, i.e. $N_i(g) = \{j : g_{ij} = 1\}$

k-Neighbourhood
: *k-neighbourhood* includes all nodes that can be reached within k steps

Degree
: *Degree* of a node i labelled $d_i(g)$ is the number of neighbours $d_i(g) = \#\{j : g_{ij} = 1\}$. When dealing with directed links, we distinguish *in-degree* and *out-degree*.

Connectivity

Global connectivity

Connected subgraph: A *connected subgraph* of a network g is a set of nodes such that for each pair of nodes there is a path between them.

Component: A *component* (N', g') of a network is a maximal connected subgraph, i.e.

(i) (N', g') is connected, and
(ii) if $i \in N'$ and $ij \in g$, then $j \in N'$ and $ij \in g'$

Connectivity

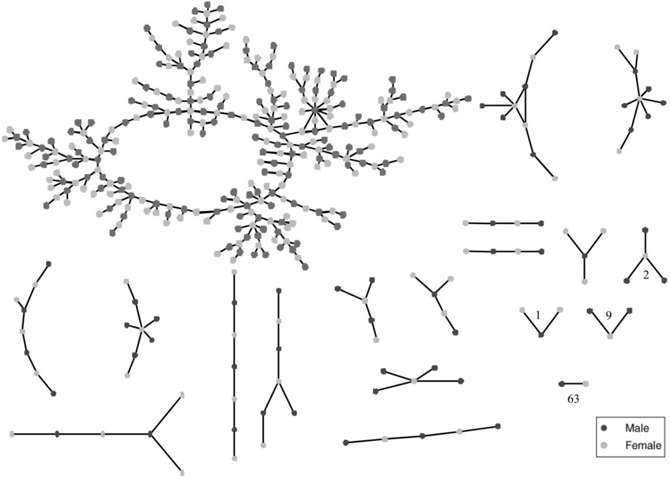


Figure: Romantic or Sexual Relationship Network; Bearman et al (2004)

Outline

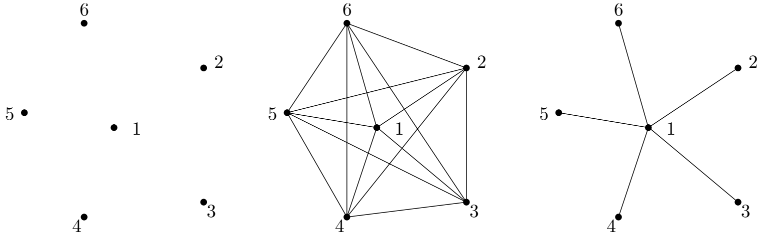
Standard networks

Empty network The *empty network* is a network without any links.

Complete The *complete network* is a network with all links in place.

Star The *star network* is a network with one central node (hub) which is involved in all links. The hub is linked to all the remaining $n-1$ nodes (spokes). The spokes are not linked with each other.

Standard networks



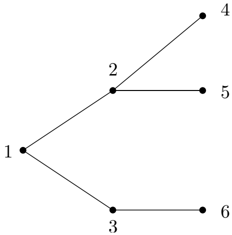
(a) Empty network (b) Complete network (c) Star network

Standard networks

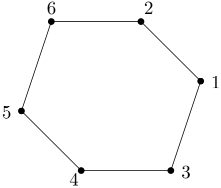
Tree A *tree network* has no cycles.

Circle A *circle* is a network with exactly one cycle and in which each node as two neighbours.

Standard networks



(a) Tree network



(b) Circle

Outline

Measuring networks

Degree based measures

Density Average degree divided by $n-1$. Measures the share of all possible links in place.

Degree distribution A function $P(d)$ which for each degree value d gives the share of nodes that have this degree.

Regular networks Networks in which all nodes have the same degree. Regular of degree k implies $P(k) = 1$ and $P(d) = 0 \, \forall d \neq k$.

Standard degree distributions

Poisson Feature of canonical random network formation, in which each link forms with probability p.
$P(d) = \binom{n-1}{d} p^d (1-p)^{n-1-d}$

Scale-free Given by $P(d) = cd^{-\gamma}$. Also called *power law*.

- Relative change in frequency when multiplying degree by factor k is $k^{-\gamma}$
- This is invariant to starting degree, i.e. at which scale the comparison is applied ("scale free")
- log-log plots show straight line

Standard degree distributions

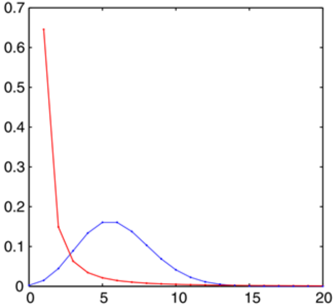


Figure: Poisson (blue, with peak) and Scale-free (red) distribution; Semitiel-Garcia et al (2012)

Scale-free networks

Real-world distributions

- Scale-free distributions have "fat tails": a relatively high frequency of nodes with very large degrees
- Such tails are often found in real world networks
  $\Rightarrow$ Many networks seen as scale-free
- But uncertainty in some cases whether other distributions might be better

Measuring networks

Distance based measures

Distance Generally refers to geodesics

Diameter Longest geodesic between any pair of nodes

Average path length Average geodesic across all pairs of nodes

Measuring networks - Small worlds

Small world property

- Idea that large networks tend to have small diameters and small average path length
- Milgram (1967) experiment:
  - Subjects in Kansas and Nebraska were told to route a letter to another unknown person in Massachusetts
  - Not directly known to subjects but name, profession, and some approximate residential details were given
  - Subjects asked to pass the letter on to someone they knew and would be likely to know the target or to be able to pass it on to someone else who did, etc.
  - Results: quarter of letters arrived at target; median number of steps was 5

Measuring networks - Small worlds

Other contexts with small world properties

- actors starring in a movie together (Watts & Strogatz, 1998)
- coauthorship in scientific journals in various fields (Newman, 2004)
- Adamic (1999) analyzes a sample of 157,127 web sites. Connecting paths existed in 85.4% with average geodesic of length 3.1

Measuring networks - Clustering

Many networks show high degree of “local cohesiveness”, clustering

Cliques
: A *clique* is a maximal subnetwork that is complete

Clustering
: Refers to “closed triangles”: if two nodes share a common neighbour, how likely is it that they are also linked?

Measuring networks - Clustering

Clustering measures

Overall clustering: Compute overall share of closed triangles across entire network

$$Cl(g) = \frac{\sum_i \#\{jk \in g \mid k \neq j, j \in N_i(g), k \in N_i(g)\}}{\sum_i \#\{jk \mid k \neq j, j \in N_i(g), k \in N_i(g)\}} \quad (1)$$

$$= \frac{\sum_{i;j\neq i;k\neq j;k\neq i} g_{ij} g_{ik} g_{jk}}{\sum_{i;j\neq i;k\neq j;k\neq i} g_{ij} g_{ik}} \quad (2)$$

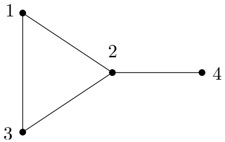
Individual clustering: Compute triangles at individual level

$$Cl_i(g) = \frac{\#\{jk \in g \mid k \neq j, j \in N_i(g), k \in N_i(g)\}}{\#\{jk \mid k \neq j, j \in N_i(g), k \in N_i(g)\}} \quad (3)$$

$$= \frac{\sum_{j\neq i;k\neq j;k\neq i} g_{ij} g_{ik} g_{jk}}{\sum_{j\neq i;k\neq j;k\neq i} g_{ij} g_{ik}} \quad (4)$$

Average clustering: $Cl^{Avg}(g) = \sum_i Cl_i(g)/n$

Measuring networks - Clustering



Node	Poss. Δ	Closed Δ	$Cl_i(g)$
1	1	1	1
2	3	1	$\frac{1}{3}$
3	1	1	1
4	0	0	0
Total	5	3	

- Overall clustering: $Cl(g) = \frac{3}{5} = 0.6$
- Average clustering:
  $Cl^{Avg}(g) = \frac{2.33}{4} = 0.5825$

Measuring networks - Clustering

Clustering in real-world networks

- Many networks show clustering coefficients much higher than predicted by random link generation
- Examples in co-authorship network studies:
  - Newman (2003): Overall clustering 0.45 in physics (random network: 0.00018)
  - Goyal et al (2006): Economics journals show clustering coefficient of 0.193 (random network: 0.000026)
  - Similar results for movie actors and for web pages

Measuring networks - Centrality

Measures of centrality

Many ways of measuring how important, powerful a node is in its network. Here some simple ones:

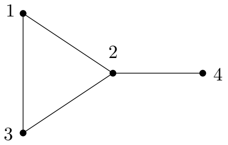
Degree Centrality: Degree centrality of i is $\frac{d_i(g)}{(n-1)} \in [0,1]$

Closeness Centrality: Closeness centrality for i rates how "far away" nodes are by using a decay factor δ for each step: $\sum_{j \neq i} \delta^{l(i,j)}$ where $l(i,j)$ is the length of the geodesic between i and j

Betweenness: Counts the share of geodesics between *other* nodes on which i is situated (proxy for intermediation power?)

$$Ce_i^B(g) = \sum_{k \neq j : i \notin \{k,j\}} \frac{P_i(kj)/P(kj)}{(n-1)(n-2)/2} \tag{5}$$

Measuring networks - Centrality



Node	Degree. Centrality	Closeness $\delta = .8$	Betweenness
1	$\frac{2}{3}$	2.24	0
2	1	2.4	$\frac{2}{3}$
3	$\frac{2}{3}$	2.24	0
4	$\frac{1}{3}$	2.08	0

- Overall clustering: $Cl(g) = \frac{3}{5} = 0.6$
- Average clustering: $Cl^{Avg}(g) = \frac{2.33}{4} = 0.5825$

Measuring networks - Centrality

- More intricate measures based on the idea that an "important" node is important because it is close to other important nodes
- Katz (1953) provides key notions and labels these "prestige"

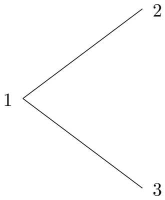
Eigenvector centrality

- Centrality $C_i^e(g)$ of a node is proportional to the total centrality of neighbours:

$$\lambda C_i^e(g) = \sum_j g_{ij} C_j^e(g) \tag{6}$$

- Can be written in matrix form as $\lambda C^e(g) = gC^e(g)$
  $\Rightarrow C^e(g)$ is eigenvector of g with eigenvalue λ
- Katz prestige is a weighted version of this measure where each g_{ij} is weighted by degree $d_j(g)$

Measuring networks - Centrality



Eigenvector centrality - Example

- Adjacency matrix:

$$g = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

- Eigenvector associated with largest eigenvalue $\lambda = 1.4142$:

$$C^e(g) = \begin{pmatrix} 0.7071 \\ 0.5000 \\ 0.5000 \end{pmatrix}$$

Measuring networks - Centrality

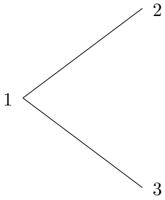
Bonacich centrality

- Based on counting number of walks in the network the node is on
- Compute paths of length k done by taking k-th power of g and multiplying by unit vector $\mathbf{1}$
- Add walks using decay parameter b and value assigned to each node a:

$$\begin{aligned} Ce^B(g,a,b) &= ag\mathbf{1} + ag \cdot bg\mathbf{1} + agb^2g^2\mathbf{1} + \ldots \\ &= (1 + bg + (bg)^2 + (bg)^3 + \ldots) \cdot ag\mathbf{1} \end{aligned}$$

- With $b = a$ this is maps into an second measure of Katz prestige

Measuring networks - Centrality



Bonacich centrality - Example

- Powers of adjacency matrix:

$$g^2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad g^3 = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

- Bonacich centrality vector with $b = .25$ and $a = 1$:

$$Ce^B(g, 1, .25) = \begin{pmatrix} 2.8571 \\ 1.7143 \\ 1.7143 \end{pmatrix}$$

Measuring networks

Correlations and Assortativity

Assortativity Refers to the correlation in degree between connected nodes. Positive assortativity implies that high-degree nodes tend to be linked to other high-degree nodes.

Homophily Refers to tendency of nodes that are similar (age, race, gender, profession, etc.) to connect

Correlations and Assortativity

Figure: Nodes coloured by student race; Currarini et al (2004)

Summary of Network Properties

Key Properties of Many Real World Networks

1. Connectedness (one or few components)
2. Small diameter
3. High Clustering
4. Heavy Tailed Degree Distribution

How to explain these properties?

Network Formation

1. Random Network Formation
2. Strategic Network Formation

Outline

Modelling Network Formation

Models of Network Formation

- Objective is to understand process underlying observed network structures
- Two main approaches:
  - Random network formation
  - Strategic network formation

Random Network Formation

Random Networks

- Based on a probabilistic process or algorithm
- Object of study is a *distribution* over possible networks
- Useful results derived using statistical analysis, generally for $n \to \infty$
- Here introduction to basic models:
  (i) Poisson random networks
  (ii) Small Worlds
  (iii) Scale free networks

Poisson Random Networks

Model of Poisson Random Networks

- Also known as Erdös & Reny networks
- Set of nodes $N = \{1, 2, \ldots, n\}$
- Each link *ij* created *independently* with probability p
- Degree distribution:

$$P(d) = \binom{n-1}{d} p^d \left(1-p\right)^{n-1-d} \tag{7}$$

- As $n \to \infty$, this is approximated by:

$$P(d) = \frac{1}{d} e^{-z} z^d \tag{8}$$

where $z = pn$, i.e. average connectivity of the network

Poisson Random Networks

Properties of Poisson Random Networks

- Useful results based on *threshold functions* $t(n)$ for $p(n)$ (p normalised as n changes, e.g. to maintain average degree)
- For $n \to \infty$, if $\frac{p(n)}{t(n)} \to \infty$, then Property P holds with high probability
- Here:
  - Connectedness: $t(n) = \frac{ln(n)}{n}$
  - Existence of Cycles and a Giant Component: $t(n) = \frac{1}{n}$

Small Worlds

A Small World Model

- Simple model of network formation generating small world properties from Watts & Strogatz (1998)
- Combines random network ideas with a regular lattice network

Small Worlds

A Small World Model

- Start with a one dimensional lattice, a ring of n vertices each connected to k nearest neighbours
- Going around the ring and selecting links to neighbours increasingly "further away", rewire each link with probability p to another node chosen uniformly
- Result of this process for intermediate p is a Small World with both (a) high clustering and (b) very short path lengths

Small Worlds

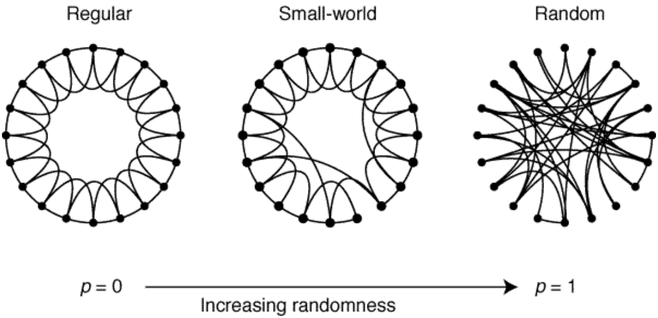


Figure: The rewiring model of Watts & Strogatz (1998)

Scale Free Networks

Model of Scale Free Networks (Barabasi & Albert, 1999)

- Two key components:
  - (i) growth
  - (ii) preferential attachment
- Both required for the model to generate scale-free networks

Scale Free Networks

Model of Scale Free Networks (Barabasi & Albert, 1999)

- Start at $t = 1$ with 2 connected nodes
- Then at each $t > 1$, add one node and connect it to one existing node
- For each existing node, probability of being connected to is proportional to its degree
  $\Rightarrow$ Degree distribution for large system:

$$P(d) = 2d^{-3} \tag{9}$$

- Simple system for generating scale free distributions

Outline

Strategic network formation

- Random network formation models generate good match for some aspects of observed networks
- But what are the *incentives* behind link formation?
- Strategic network formation explicitly takes economic approach
- Distinguish:
  - Stability: given g, are there incentives for change?
  - Equilibrium: g is created by actions selected by players in games of network formation
- Also introduces explicit view on utility from networks $u_i(g)$ and thus efficiency

Strategic Network Formation – Efficiency

Definition (Efficient Networks)

A network g is *efficient* relative to a profile of utility functions $(u_1, \ldots, u_n)$ if

$$\sum_i u_i(g) \geq \sum_i u_i(g') \text{ for all } g' \in G(N) \tag{10}$$

Can also consider *Pareto efficiency* in the usual sense

Strategic Network Formation – Pairwise Stability

	Link	Don't
Link	1, 1	0, 0
Don't	0, 0	0, 0

Pairwise Stability – Why a new concept?

- Links connect *two* agents – Need two agents to create one?
- Simple game: 2 players decide whether to form a link or not; payoff of 1 if both agree to form
- Two pure strategy Nash equilibria: (Link, Link), (Don't, Don't)
- Should players be able to move away from "bad" equilibrium?

Strategic Network Formation – Pairwise Stability

Definition (Pairwise stability (Jackson & Wolinsky, 1996))

A network g is *pairwise stable* if

1. for all $ij \in g$, $u_i(g) \geq u_i(g - ij)$ and $u_j(g) \geq u_j(g - ij)$, and
2. for all $ij \notin g$, if $u_i(g + ij) > u_i(g)$ then $u_j(g + ij) < u_j(g)$.

Checks for unilateral individual link destruction and bilateral individual link creation

Strategic Network Formation – Illustration

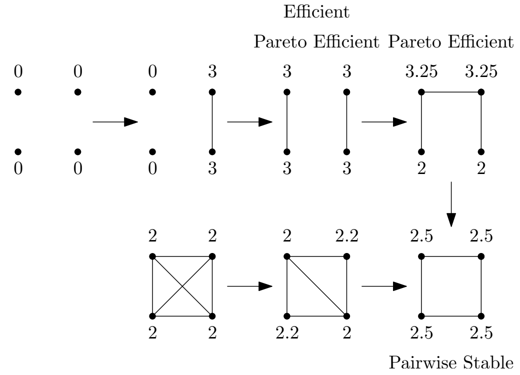


Figure: A simple illustration with four players, Jackson (2008)

Strategic Network Formation – Distance Based Utility

The Distance Based Utility Model

- Distance-based benefit derived from all nodes connected to:

$$u_i(g) = \sum_{j \neq i \in N} \delta^{l_{ij}(g)} - d_i(g)c$$

- Benefit decays with distance $l_{ij}(g)$
- Each additional link in degree $d_i(g)$ adds cost c

Strategic Network Formation – Distance Based Utility

Proposition (Efficient networks (Jackson & Wolinsky, 1996))

The unique efficient network structure is

(i) *The complete network if* $c < \delta - \delta^2$

(ii) *A star encompassing all nodes if* $\delta - \delta^2 < c < \delta + \frac{n-2}{2}\delta^2$

(iii) *The empty network if* $\delta + \frac{n-2}{2}\delta^2 < c$

Strategic Network Formation – Distance Based Utility

Proof of (i): Complete if $c < \delta - \delta^2$

- By contradiction. Assume g not complete and efficient.
- Now select a pair of nodes i and j not connected.
- Adding link ij cannot decrease the utility of any $k \notin \{i, j\}$.
- Adding link ij cannot increase distances between i and j and any other nodes
- Adding link ij decreases the distance between i and j (to one step) which yields net benefit for each of at least $\delta - c - \delta^2 > 0$ (by parameter assumption)
- Thus, adding ij increases total benefit, yielding a contradiction.

Strategic Network Formation – Distance Based Utility

Proof of (ii): Star if $\delta - \delta^2 < c < \delta + \frac{n-2}{2}\delta^2$

- Connecting k nodes involves *at least* $k-1$ links
- A star network involves exactly $k-1$ links and yields utility:

$$2(k-1)(\delta - c) + (k-1)(k-2)\delta^2 \tag{11}$$

- Consider some component of k nodes with $m \geq k-1$ links. This yields at most:

$$2m(\delta - c) + 2\left[\frac{k(k-1)}{2} - m\right]\delta^2 \tag{12}$$

- (11) - (12) yields:

$$2\left[m - (k-1)\right]\left[\delta^2 - (\delta - c)\right]$$

$\Rightarrow$ Optimal $m = k-1$

Strategic Network Formation – Distance Based Utility

Proof of (ii) ctd: Star if $\delta - \delta^2 < c < \delta + \frac{n-2}{2}\delta^2$

- Consider now networks with $k-1$ links
- Any network with $k-1$ links connecting k nodes that is not a star has at least one pair of nodes at distance > 2
- Thus total utility is

$$
\begin{aligned}
&2(k-1)(\delta - c) + X \\
<&2(k-1)(\delta - c) + (k-1)(k-2)\delta^2
\end{aligned}
$$

$\Rightarrow$ Efficient networks consist of stars and disconnected nodes

Strategic Network Formation – Distance Based Utility

Proof of (ii) ctd: Star if $\delta - \delta^2 < c < \delta + \frac{n-2}{2}\delta^2$

- Now, assume there are two stars (with $k_1 \geq 1$ and $k_2 \geq 2$ nodes, respectively) with nonnegative utility
- A *combined* single star yields higher total utility:

$$
\begin{aligned}
& (k_1 + k_2 - 1)\left[2(\delta - c) + (k_1 + k_2 - 2)\delta^2\right] \\
> \quad & (k_1 - 1)[2(\delta - c) + (k_1 - 2)\delta^2] \\
+ \quad & (k_2 - 1)[2(\delta - c) + (k_2 - 2)\delta^2]
\end{aligned}
$$

  $\Rightarrow$ If $\delta^2 > \delta - c$, efficient network is either a star involving all nodes ($k = n$) or the empty network
- If $c < \delta + \frac{n-2}{2}\delta^2$, then star gives positive utility
- Also gives part (iii)

Strategic Network Formation – Distance Based Utility

Proposition (Pairwise stable networks (Jackson & Wolinsky, 1996))

(i) *A pairwise stable network has at most one (nonempty) component.*

(ii) *For $c < \delta - \delta^2$, the unique pairwise stable network is the complete network.*

(iii) *For $\delta - \delta^2 < c < \delta$ a star encompassing all players is pairwise stable, but for some n and parameter values in this range it is not the unique pairwise stable network.*

(iv) *For $\delta < c$, in any pairwise stable network each node has either no links or else at least two links.*

Strategic Network Formation – Distance Based Utility

Part (i) - Proof

- Proof by contradiction. Assume there exist two (nonempty) components and the network is pairwise stable.
- Consider i in one component and j connected to k in another component
- Utility to i from linking to k is at least as large as j's marginal utility from linking to k *plus* the value of an indirect connection to j
  $\Rightarrow$ At least $u_j(g) - u_j(g - jk) + \delta^2$
- This is larger than the marginal value of the link jk to j which is nonnegative since j does not wish to delete it
- Similarly, k sees an increase in payoffs from link ik
  $\Rightarrow$ i and k would benefit from ik. Contradicts pairwise stability.

Strategic Network Formation – Distance Based Utility

Part (ii) - Proof

- Again by contradiction. Assume a network that is not complete and pairwise stable.
- Take a pair i, j that is not connected and consider adding link ij
- Payoff to i and j is at least $\delta - \delta^2 - c > 0$
  $\Rightarrow$ i and j would benefit from ij. Contradicts pairwise stability.

Strategic Network Formation – Distance Based Utility

Part (iii) - Proof

- Link destruction: As $c < \delta$ no player wants to delete any link in a star
- Link creation: As $\delta - \delta^2 < c$ no two peripheral players wish to add a link
  $\Rightarrow$ Star is pairwise stable
- Second part by example: A circle of 4 nodes is pairwise stable if $\delta - \delta^2 < c < \delta - \delta^3$

Strategic Network Formation – Distance Based Utility

Part (iv) - Proof

- By contradiction. Assume network g with $d_i(g) = 1$ and pairwise stable.
- If $c > \delta$, $u_i(g - ij) - u_i(g) = c - \delta > 0$
  $\Rightarrow$ i benefits from destroying link. Contradicts pairwise stability.

Strategic Network Formation – Distance Based Utility

Stability vs. Efficiency

- Discrepancy between stability and efficiency
- Source: Externalities
- Here: Link generates indirect connections for others
- To overcome, need conditional contracts / transfers

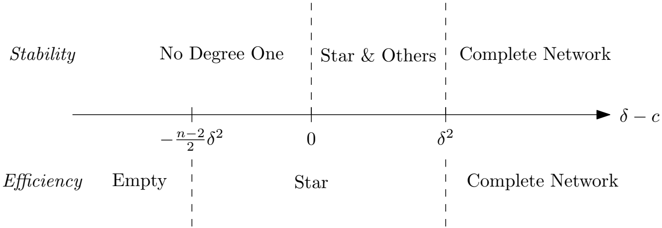


Figure: Efficiency vs. Pairwise Stability in the Connections Model

Strategic Network Formation – Link Announcement Game

Network Formation Games

- Previous analysis based on “stability”
- Alternatively will consider full game of link formation
- Focus on use of different equilibrium notions

Strategic Network Formation – Link Announcement Game

The Link Announcement Game (Myerson, 1977)

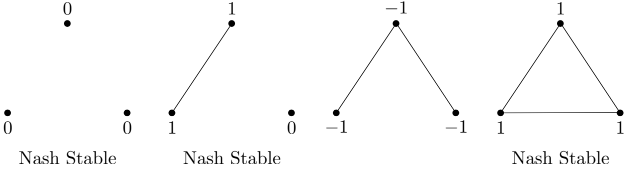
- Each player announces at the beginning which links she would like to form
- A link is formed if both players involved announce this link
- Utility is derived from network
- Formally:
  - Strategy space: $S_i = 2^{N \setminus i}$
  - Strategy profile: $s \in S_1 \times S_1 \times \ldots \times S_N$
  - Resulting network: $g(s) = \{ij| \mid i \in s_j \text{and} j \in s_i\}$
  - Payoffs: $u_i(g)$

Strategic Network Formation – Stability Notions

Definition (Nash Stability)

Network is *Nash stable* if it results from a Nash equilibrium strategy profile

- Too many equilibria? Empty network is always Nash stable.

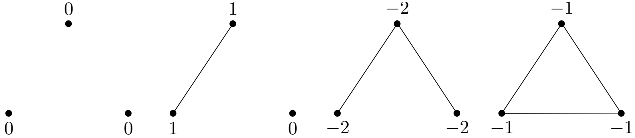


Strategic Network Formation – Stability Notions

Definition (Pairwise Nash Stability)

Network is *pairwise Nash stable* if it is Nash stable and pairwise stable

- Difference to pairwise stability: deletion of multiple links



Nash Stable   Pairwise & Nash Stable   Pairwise Stable

Strategic Network Formation – Stability Notions

Definition (Strong Stability)

Network is *strongly stable* if there is no coalition that can profitably deviate to another network that is not worse for at least one agent in the coalition

- Strong stability implies Pareto efficiency

Strategic Network Formation – Distance Based Utility

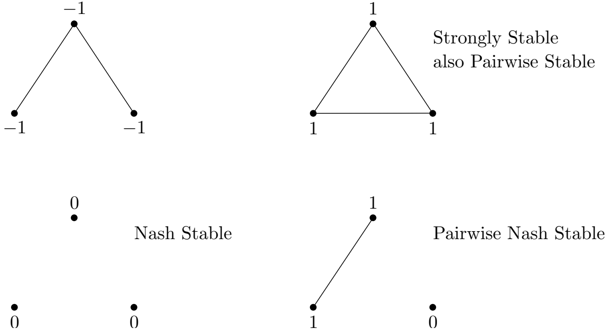


Figure: Comparison of Stability Notions (Jackson, 2009)